7. TRIGONOMETRIC FUNCTIONS

§7.1. Elementary Trigonometry

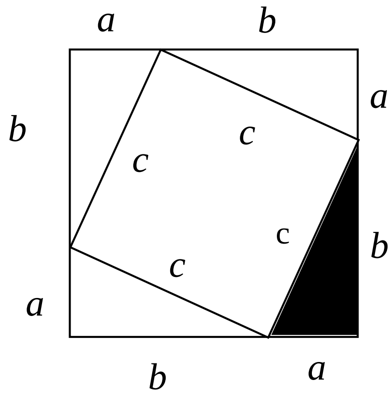
In this section we revise the elementary trigonometry that we learn in High School. It concerns angles and triangles, especially right-angled ones. Angles are traditionally measured in degrees, where 360 degrees constitute a complete revolution. This is a legacy from the Babylonians who used a number system based on base 60. The other measure that comes from the Babylonians is time where there are 60 minutes in an hour and 60 seconds in a minute.

A perpendicular angle, or a **right angle**, has 90 degrees. We write this as 90°. A right-angled triangle is one where one angle is a right angle. The other two angles add up to 90° and are called **complementary angles**.

The longest side of a right-angled triangle, the side opposite the right angle, is called the **hypotenuse**. The fundamental fact about right-angled triangles is the theorem that is attributed to Pythagoras: The square on the hypotenuse is the sum of the squares on the other two sides.

Theorem 1 (Pythagoras): If the lengths of the sides of a right-angled triangle are a, b, c and c is the length of the hypotenuse then $a^2 + b^2 = c^2$.

Proof:

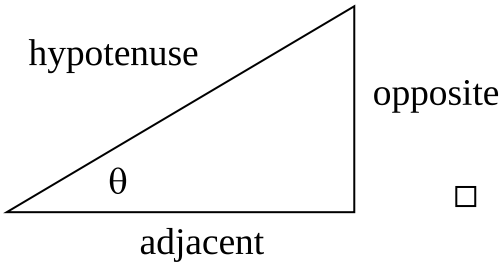


The areas of the larger square is:

$$(a + b)^2 = c^2 + 4.\ ½\ .ab = c^2 + 2ab.$$

Hence $a^2 + b^2 = c^2$. 🖐☺

Suppose we have a right-angled triangle where the angles are 90, θ and 90 – θ.



If we focus on the angle θ then the **opposite side** is, as the name suggests, the side opposite that angle. The **hypotenuse** is the side opposite the right-angle and the remaining side is called the **adjacent side**.

The sides of the triangle will change as the triangle gets bigger or smaller, with the same angle θ. But, since corresponding sides of similar triangles are proportional the ratio of corresponding sides will depend only on the angle.

We define the **sine** of θ as $\mathbf{\sin\theta = \dfrac{opposite}{hypotenuse}}$,

the **cosine** of θ is defined as $\mathbf{\cos\theta = \dfrac{adjacent}{hypotenuse}}$ and

the **tangent** of θ is defined as $\mathbf{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$ = $\dfrac{\text{opposite}}{\text{adjacent}}$.

A consequence of Pythagoras' Theorem is that: $\mathbf{\sin^2\theta + \cos^2\theta = 1}$**.**

[Here we write $\sin^2\theta$ when we mean $(\sin\theta)^2$. Similar conventions apply to $\cos^2\theta$ and $\tan^2\theta$.]

The three other rations are the reciprocals of the above.

The **secant** of θ is $\mathbf{\sec\theta = \dfrac{adjacent}{hypotenuse}}$,

the **cosecant** of θ is $\mathbf{\mathrm{cosec}\,\theta = \dfrac{hypotenuse}{opposite}}$ and

the **cotangent** of θ is $\mathbf{\cot\theta = \dfrac{adjacent}{opposite}}$.

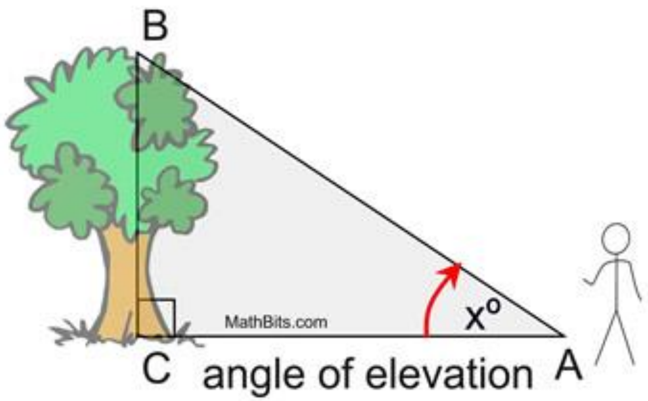
Rather than burden our brains with these definitions we remember that

$$\mathbf{sec\theta = \frac{1}{cos\theta}},\ \mathbf{cosec\theta = \frac{1}{sin\theta}}\ \ \mathbf{and\ cot\theta = \frac{1}{tan\theta}}\ .$$

Simple consequences of these definitions are the trigonometric identities:

$\mathbf{sec^2\theta = 1 + tan^2\theta}$ and $\mathbf{cosec^2 = 1 + cot^2\theta}$.

The first of these is important enough to be memorised but the second is not. It can always be worked out when needed.

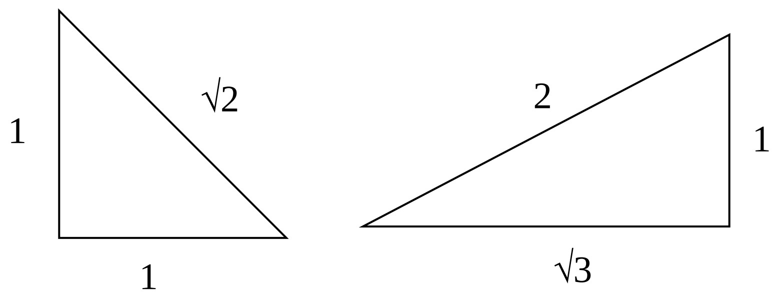


A general principle with an area of mathematics, like trigonometry, with many formulae is to remember only a key few and to be able to work out any others that might be needed as the occasion arises.

Basic Trigonometry is used in surveying. For example if we measure the angle of elevation of a tree, and the distance to the base of a tree, we can work out the height of the tree using the tan function. However Trigonometry is used in many applications where there are no angles or lengths. For example, in electronics, one of the basic ingredients is the sine wave,

§7.2. Special Angles

There are two right-angled triangles that are rather special: the 45 degree triangle and the 60-30 triangle. Indeed these are the two standard shapes for set squares.



The dimensions can be obtained from the fact that a 45 degree right-angled triangle is isosceles, a 60-30 triangle is half an equilateral triangle and, of course, Pythagoras' Theorem. Although a right-angled triangle with two right angles and one zero angle is not a proper triangle we can infer the trigonometric functions for 0° and 90°. These are included in the following table.

θ	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

The last value is a bit of a cheat because there is no such number as infinity. It reflects the fact that if θ is very close to 90° the adjacent side is very short and so tan θ is very large but in fact tan 90 does not exist. We can make tan θ as large as we like by making θ sufficiently close to 90°. Usually we capture this property by saying that as θ approaches 90°, tan θ approaches infinity.

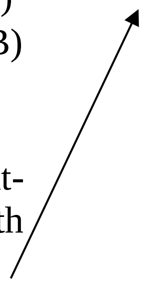
§7.3. Sum and Difference of Angles

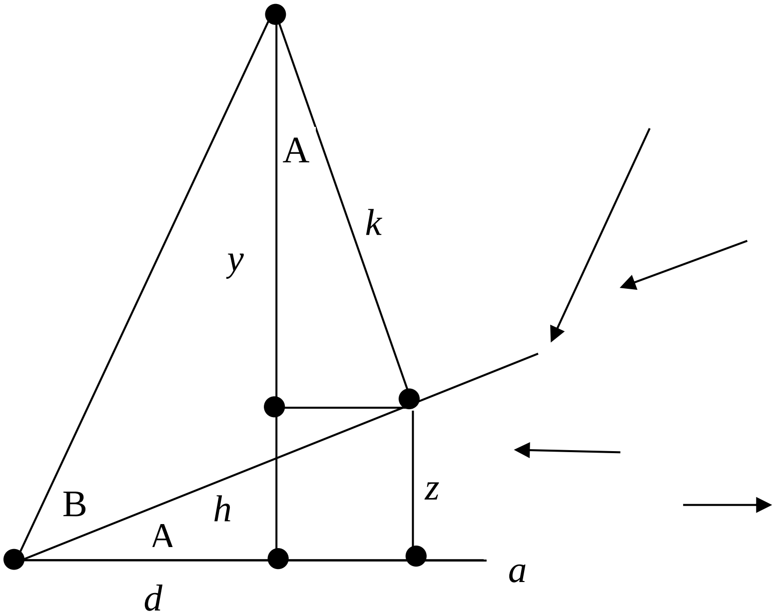
Theorem 2: sin(A + B) = sin(A) cos(B) + cos(A) sin(B)

cos(A + B) = cos(A) cos(B) – cos(A) cos(B)

$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}.$$

Proof: In the following diagram we have four right-angled triangles, two with angles A, and one each with angles B and A + B.





$$\sin(A + B) = \frac{z + y}{d} = \frac{z}{h} \cdot \frac{h}{d} + \frac{y}{k} \cdot \frac{k}{d}$$

$$= \sin(A)\cos(B) + \cos(A)\sin(B).$$

$$\cos(A + B) = \frac{a}{d} = \frac{(a + b) - b}{d} = \frac{(a + b)}{h} \cdot \frac{h}{d} - \frac{b}{k} \cdot \frac{k}{d}$$

$$= \cos(A)\cos(B) - \sin(A)\sin(B).$$

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$= \frac{\sin(A)\cos(B) + \cos(A)\sin(B)}{\cos(A)\cos(B) - \sin(A)\sin(B)}$$

$$= \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$ after dividing top and bottom by cos(A) cos(B). ✋☺

Corollary: sin(2A) = 2sin(A) cos(A),

$\cos(2A) = \cos^2(A) - \sin^2(A)$ and

$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}.$$

The above identities are usually called the **Double Angle Formulae**. The following are the **Half Angle Formulae** where sin(A), cos(A) and tan(A) are all expressed in terms of t = tan(A/2).

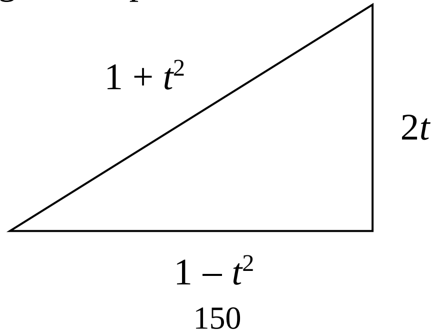
Theorem 3: If t = tan(A/2) then

$$\sin(A) = \frac{2t}{1 + t^2},$$

$$\cos(A) = \frac{1 - t^2}{1 + t^2} \text{ and}$$

$$\tan(A) = \frac{2t}{1 - t^2}.$$

Proof: The third of these is simply the Double Angle Formula, using A/2 in place of A.



By drawing a suitable right-angled triangle, with sides $2t$ and $1 - t^2$ we can deduce that the hypotenuse is $1 + t^2$ and hence read off sin(A) and cos(A). ✋☺

Theorem 4: $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$,

$$\cos(A - B) = \cos(A)\cos(B) + \cos(A)\cos(B)$$

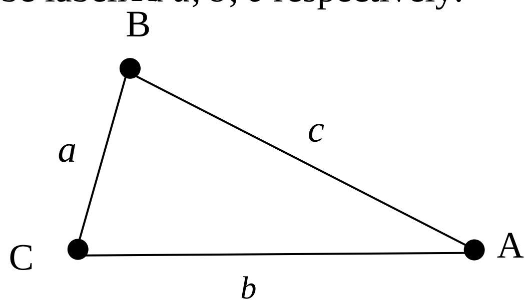
and

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}.$$

Proof: This is proved similarly using the diagram from Theorem 2. Let C = A + B. Then by similar methods to those used in the proof of Theorem 2 we get:
$\sin(C - B) = \sin(C)\cos(B) - \cos(C)\sin(B)$ which is what we have to prove but with different notation. The proof of the other identities is left as an exercise. ✋☺

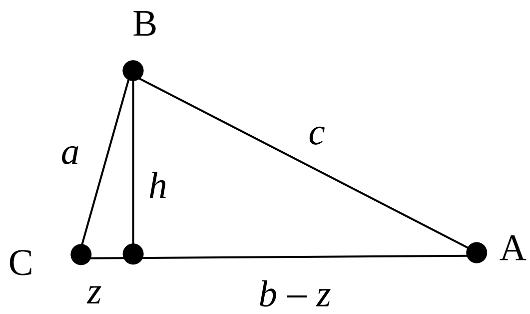
§7.4. Completing Triangles

We now move away from right-angled triangles to consider general triangles. We'll label the three vertices A, B, C and also use the symbols A, B, C for the values of the corresponding angles. The sides opposite these sides will be labelled *a*, *b*, *c* respectively.



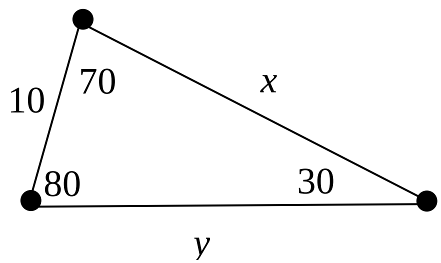
Theorem 5 (Sine Rule): $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$.

Proof: Drop a perpendicular from A to the side BC.



Then $h = a.\sin C = c.\sin A$. By drawing perpendiculars to the other two sides the result follows. 🖐☺

Example 1: Find the remaining three sides of the following triangle



Solution: $\frac{x}{\sin 80} = \frac{10}{\sin 30} = \frac{10}{½} = 20.$

Hence $x = 20 \sin 80 = 20 \times 0.9848 = 19.6960$ and

$\frac{y}{\sin 70} = 20$ so $y = 20 \sin 70 = 20\times0.9397 = 18.7940$.

Theorem 6 (Cosine Rule): $c^2 = a^2 + b^2 - 2ab.\cos C$.
Proof: $z = a.\cos C$ and $h = a.\sin C$.
Hence, by Pythagoras in the right triangle:

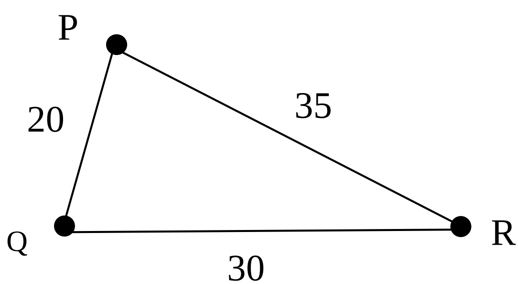
$$c^2 = (b - a.\cos C)^2 + a^2\sin^2 C$$
$$= b^2 + a^2\cos^2 C - 2ab.\cos C + a^2\sin^2 C$$
$$= a^2 + b^2 - 2ab.\cos C.$$

This can be used to find the angles of a triangle if we are just given the sides:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

It's probably best to remember the Cosine Rule as "the square on one side is the sum of the squares on the other two sides minus twice the product of the other two sides and the cosine of the included angle". This way, if the notation varies you can easily adapt the cosine rule.

Example 2: Find the three angles of the following triangle:



Solution: $\cos P = \dfrac{20^2 + 35^2 - 30^2}{2.20.35} = \dfrac{725}{1400} = 0.5179.$

Hence P = 58.8°.

$\cos Q = \dfrac{20^2 + 30^2 - 35^2}{2.20.30} = \dfrac{75}{1200} = 0.0625.$ Q = 86.4°.

Hence R = 180° – 58.8° – 86.4° = 34.8°.

(Of course we could have used the cosine rule one more time to find R and use this as a check.) If you don't know how to get from cos P = 0.5179 to P = 58.8° using your calculator look ahead to §7.7.

These formulae are useful, in many cases, for finding the remaining sides and angles given three pieces of information. If we're given two angles, we know the third angle, and if we know one side we can use the Sine Rule to obtain the other two sides.

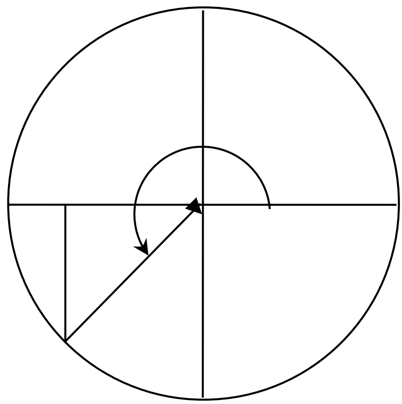
If we're given two sides and the included angle we can use the Cosine rule to find the other side, and then the Sine Rule to find the other two angles.

If we're given two sides and a non-included angle or three angles the triangle is not uniquely defined.

If we're given three sides we can use the Cosine Rule to find the three angles.

§7.5. Rotations in the *x-y* Plane

So far angles have been restricted to the range 0° to 90°. We now extend the definition of sinθ, etc for all θ. We do this by considering the unit circle (a circle of radius 1) and rotations in the x-y plane.



We begin with the positive x-axis and measure rotations as positive in the anti-clockwise direction and negative in the clockwise direction. The rotation indicated above is positive and its magnitude is about 225°. If this rotation were to be carried out clockwise it would have been through an angle of 135°. If this clockwise rotation had been carried out twice it would have been through an angle of 270°.

If the point (1, 0), on the positive half of the x-axis, is rotated through an angle θ, the resulting point is defined to be $(\cos\theta, \sin\theta)$. This now extends the definition of $\sin\theta$ and $\cos\theta$ to all θ. For example, as the above diagram shows, if the point (1, 0) is rotated through 225° the resulting point is $(-1/\sqrt{2}, 1,\sqrt{2})$. (That's because the right-angled triangle shown has two equal sides and its hypotenuse is 1. Hence the length of each side is $1/\sqrt{2}$. It follows that $\cos 225° = -1/\sqrt{2}$ and $\sin 225° = 1/\sqrt{2}$.

If we rotated (1, 0) through 585°, that is 225° plus 360°, the resulting point would still be $(-1/\sqrt{2}, 1,\sqrt{2})$. An extra 360° in either direction doesn't make any difference to the position of the rotated point and therefore makes no difference to the values of $\sin\theta$ and $\cos\theta$.

Theorem 7: For all angles, θ, measured in degrees:
$\sin(\theta \pm 360) = \sin\theta$ and $\cos(\theta \pm 360) = \cos\theta$,
$\sin(\theta + 180) = -\sin\theta$ and $\cos(\theta + 180) = -\cos\theta$,
$\sin(\theta + 90) = \cos\theta$ and $\cos(\theta + 90) = -\sin\theta$.
Proof: These can easily be proved by considering the *x*-*y* plane. ✋

The absolute values of $\sin\theta$ and $\cos\theta$ lie between 0 and 1 but in different quadrants they have different signs. By considering the *x*-*y* plane we can see that these signs are as follows.

$\sin\theta > 0$ $\cos\theta < 0$	$\sin\theta > 0$ $\cos\theta > 0$
$\sin\theta < 0$ $\cos\theta < 0$	$\sin\theta < 0$ $\cos\theta > 0$

We define $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and so $\tan(\theta + 180) = -\tan\theta$ etc.

The value of $\tan\theta$ is positive in the 1st and 3rd quadrants and negative in the others. A useful diagram, to help you remember in which quadrants $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive is the following.

S	A
T	C

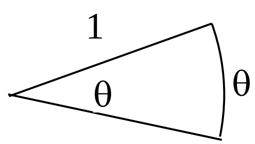
Here A stands for ALL, S stands for SIN, T stands for TAN and C stands for COS. Students are often taught a mnemonic "All Stations To ..." where C is represented by some local railway station that starts with "C". As I learnt this at Canterbury Boys High School, naturally our teachers taught us to remember "All Stations To Canterbury."

Since most of our proofs of trigonometric identities relied on diagrams where angles were in the range 0 to 90 we should provide new proofs to guarantee that they work for all angles. We won't do that because we're not yet at the final stage of our development of trigonometry.

§7.6. Radian Measure

Degrees are a very suitable unit of measure for angles. Other systems have been proposed such as the one that assigns 100 'grads' to a right-angle, but this hasn't caught on. What about using a unit that's about 57.29577951 degrees? For practical purposes it would be a nightmare. And yet that is the natural unit of measure for reasons that we'll now see. In fact, unless you're a surveyor or a navigator you'll be leaving degrees behind and adopting **radians** as the unit.

Radian measure is based on arc lengths. If we have a sector of a circle with unit radius we define the angle as the length of the arc.



Since the circumference of a circle with radius 1 is 2π, $360°$ is 2π radians and $90°$ is $\pi/2$. The special angles in radians are as follows:

degrees	radians
0	0
30	π/6
45	π/4
60	π/3
90	π/2

We choose not to evaluate these as decimals. It is easier, and more exact, to leave them as multiples of π. From now on all angles will be in radians and therefore we write:

$\sin(\theta \pm 2\pi) = \sin\theta$, $\cos(\theta \pm 2\pi) = \cos\theta$, $\tan(\theta + 2\pi) = \tan\theta$,

$\sin(\theta + \pi) = -\sin\theta$, $\cos(\theta + \pi) = -\cos\theta$, $\tan(\theta + \pi) = \tan\theta$,

$\sin(\pi/2 - \theta) = \cos\theta$.

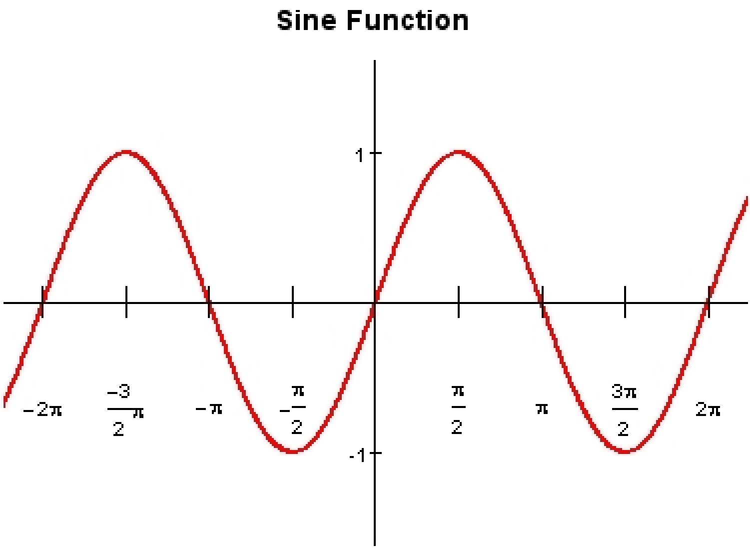
§7.7. The Functions sin *x*, cos *x* and tan *x*

Up till now we've tended to write the sine function as sin θ, or sin A. That's because we've thought of the θ or the A as an angle. Many important applications of the trigonometric functions have nothing to do with angles. So we're now going to think of sine and cosine as applying to real numbers, rather than angles. For this reason we'll tend to write them as sin *x* and cos *x*.

The values of these functions are the values that would apply if *x* was an angle measured in radians, so we're not moving too far from what we know already. It's just a point of view.

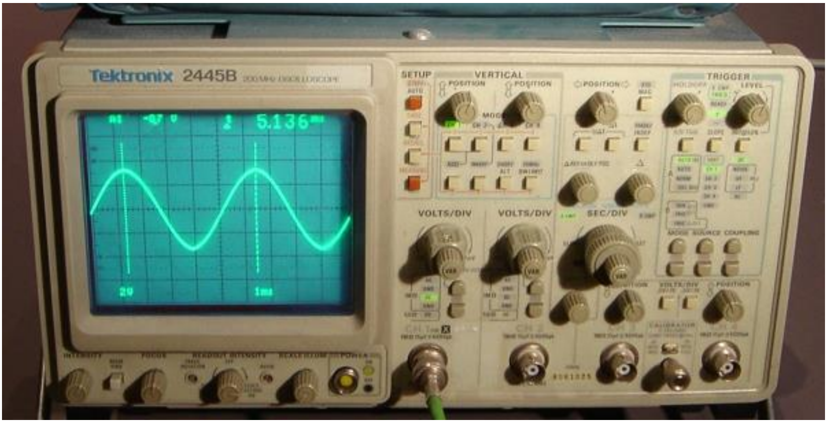
If we put our calculator into radian mode (you'll have to read the instruction book to see how this is done for your particular calculator),then enter a number and then press the SIN key you'll get the value of sin *x*. To check that you can operate your calculator correctly in this way insert 2 and press SIN. You should get approximately 0.9093. (If you get 0.0349 your calculator is in degrees mode.) You will have shown that sin 2 = 0.9093. One interpretation of this is that the sine of an angle of 2 radians (about 114°) is 0.9083. But there many other interpretations and there are many uses for the sine function other than something to do with angles.

So try to think of sin *x* in the same way as you think of $\sqrt{x}$. What might help is to see the graph of $y = \sin x$. Here the *x* is measured along the *x*-axis and so is represented by a length, not an angle.

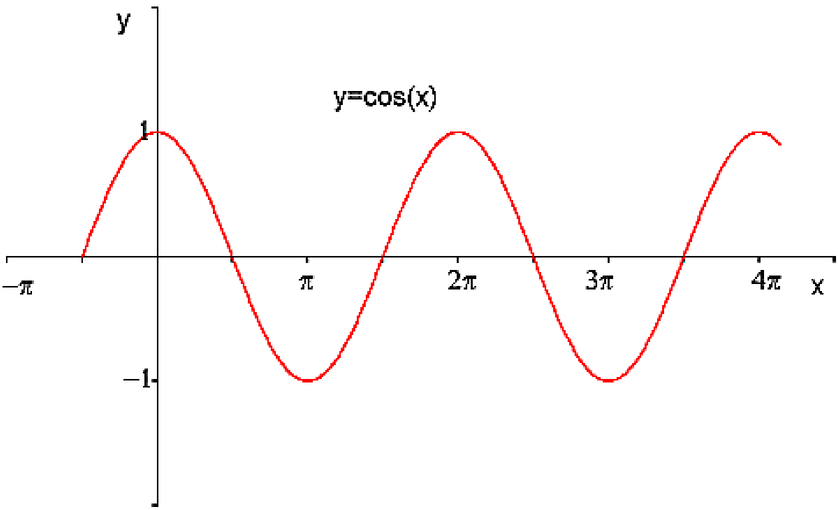


Notice all the key aspects of sin *x*. When *x* = 0, sin *x* = 0. Then as *x* increases, the value of sin *x* increases until it reaches its maximum at *x* = π/2 (about 1.57). It then starts decreasing and is again 0 when *x* = π. From *x* = π to *x* = 2π sin *x* is negative. It decreases to a minimum at *x* = 3π/2 and then works its way back to zero. From here it starts its fluctuation all over again.

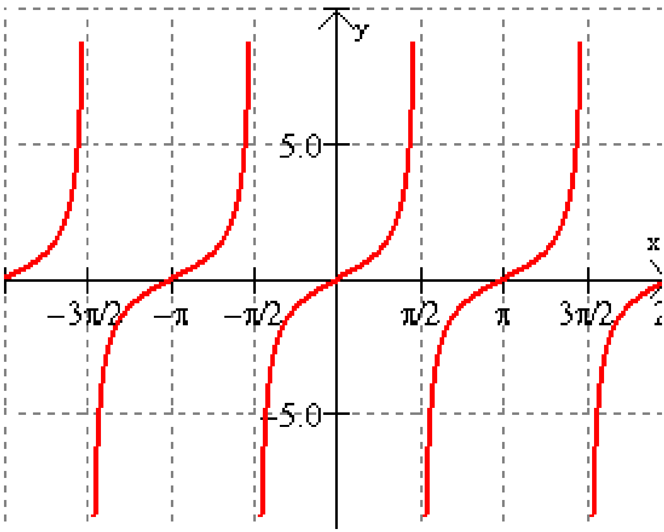
You can clearly see the periodic nature of the sine function. Also there is the fact that the graph looks the same if you turn it upside down (rotating it through 180 degrees about the *x*-axis). This is due to the fact that sin(–*x*) = – sin *x*.



The graph of $y = \cos x$ has exactly the same shape. It is just translated along the x-axis through a distance of $\pi/2$. This reflects the fact that $\cos x = \sin(x + \pi/2)$.



The graph of $y = \tan x$ looks quite different. Notice how there are discontinuities at $x = \pi/2, 3\pi/2$ etc. When we said $\tan \pi/2 = \infty$ we were only telling half the story. Because it is true that $\tan x$ approaches ∞ as x approaches $\pi/2$ *from below*. But when x approaches $\pi/2$ from above $\tan x$ approaches $-\infty$.



Whenever there is a repetitive action the sine function is likely to be close by. A pendulum on a clock, a piston moving backwards and forwards, these are examples of **simple harmonic motion**. If you have ever seen an oscilloscope (an electronic display that shows sounds as pictures) showing a pure musical tone, you will see a sine curve.